7 Networks at variable frequency

Student Group

First Name	Surname	Matrikel Nr.

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Introduction

At the previous chapters it was explained how the “influence of a sinusoidal current flow” of capacitor and inductors look like. To describe this, the impedance was introduced. This can be understood as a complex resistance for sinusoidal excitation.

It applies to the capacitor:

$$\begin{align*} \underline{U}_C = \frac{1}{j\omega \cdot C} \cdot \underline{I}_C \quad \rightarrow \quad \underline{Z}_C = \frac{1}{j\omega \cdot C} \end{align*}$$

and for the inductance

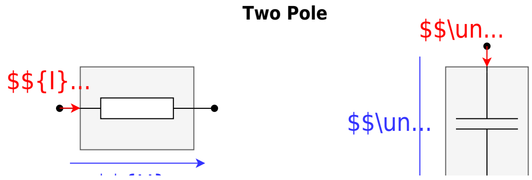
$$\begin{align*} \underline{U}_L = j\omega \cdot L \cdot \underline{I}_L \quad \rightarrow \quad \underline{Z}_L = j\omega \cdot L \end{align*}$$

Complex impedances can be dealt with in much the same way as ohmic resistances in Electrical Engineering 1 (see: simple DC Circuits, linear Sources and two-terminal network, Analysis of DC Networks). In these transformations, the fraction $j\omega \cdot$ is preserved. Circuits with impedances such as inductors and capacitors will show a frequency dependence accordingly.

> **Targets**
>
> After this lesson, you should:
>
> 1. know that …
> 2. know that … is formed.
> 3. be able to … can …

7.1 From Two-Terminal Network to Four-Terminal Network

Fig. 1: Two-Terminal Network to Four-Terminal Network



Until now, components such as resistors, capacitors and inductors have been understood as two-terminal. This is also obvious, since there are only two connections. In the following however circuits are considered, which behave similar to a voltage divider: On one side a voltage U_I is applied, on the other side U_O is formed with it. This results in 4 terminals. The circuit can and will be considered as a four-terminal network in the following. However, the input and output values will be complex.

For a four-terminal network, the relation of “what goes out” (e.g. $\underline{U}_O$ or $\underline{U}_2$) to “what goes in” (e.g. voltage $\underline{U}_I$ or $\underline{U}_1$) is important. Thus, the output and input variables ($\underline{U}_O$) and ($\underline{U}_I$) give the quotient:

$$\begin{align*} \underline{A} & = {{\underline{U}_O^}\over{\underline{U}_I^}} \\ & \text{with} \; \underline{U}_O = U_O \cdot e^{j \varphi_{uO}} \\ & \text{and} \; \underline{U}_I = U_I \cdot e^{j \varphi_{uI}} \\ \\ \underline{A} & = \frac {\underline{U}_O^}{\underline{U}_I^} = \frac {U_O \cdot e^{j \varphi_{uO}}}{U_I\cdot e^{j \varphi_{uI}}} \\ & = \frac {U_O}{U_I}\cdot e^{j (\varphi_{uO}-\varphi_{uI})} \\ \end{align*}$$

$$\begin{align*} \boxed{\underline{A} = \frac {\underline{U}_O^}{\underline{U}_I^} = \frac {U_O}{U_I}\cdot e^{j \Delta\varphi_{u}}} \end{align*}$$

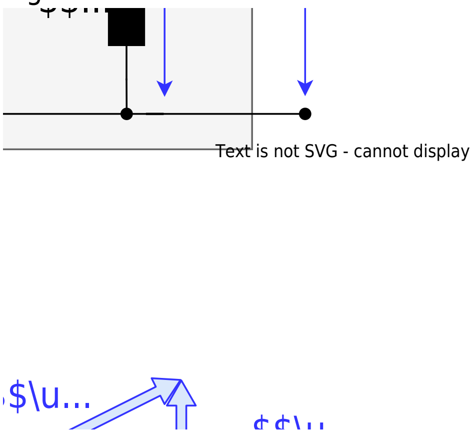
Reminder:

- The complex-valued quotient ${\underline{U}_O}/{\underline{U}_I}$ is called the **transfer function**.
- The frequency-dependent magnitude of the quotient $A(\omega)={U_O}/{U_I}$ is called **amplitude response** and the angular difference $\Delta\varphi_{u}(\omega)$ is called **phase response**.

The frequency behaviour of the amplitude response and the frequency response is not only important in electrical engineering and electronics, but will also play a central role in control engineering.

7.2 RL Series Circuit

Fig. 2: RL-series



First, a series connection of a resistor R and an inductor L shall be considered (see figure 2). This structure is also called RL-element.
Here, $\underline{U}_I= \underline{X_I} \cdot \underline{I}_I$ with $\underline{X}_I = R + j\omega \cdot L$ and corresponding for $\underline{U}_O$: \begin{align*} \underline{A} = \frac {\underline{U}_O^}{\underline{U}_I^} = \frac {\omega L}{\sqrt{R^2 + (\omega L)^2}}\cdot e^{j\left(\frac{\pi}{2} - arctan \frac{\omega L}{R} \right)} \end{align*}

This results in the following for

- the amplitude response: $A = \frac {\omega L}{\sqrt{R^2 + (\omega L)^2}}$ and
- the phase response: $\Delta\varphi_{u} = arctan \frac{R}{\omega L} = \frac{\pi}{2} - arctan \frac{\omega L}{R}$

The main focus should first be on the amplitude response. Its frequency response can be derived from the equation in various ways.

1. Extreme frequency consideration of this RL circuit (in the equation and in the system)
2. Plotting amplitude and frequency response
3. Determination of prominent frequencies

These three points are now to be gone through.

7.2.1 RL High Pass

For the first step we investigate the limit consideration: Wwe look at what happens, when the frequency ω runs to the definition range limits, i.e. $\omega \rightarrow 0$ and $\omega \rightarrow \infty$:

- For $\omega \rightarrow 0$, $A = \frac {\omega L}{\sqrt{R^2 + (\omega L)^2}} \rightarrow 0$ as the numerator approaches zero and the denominator remains greater than zero.
- For $\omega \rightarrow \infty$, $A \rightarrow1$, because in the root in the denominator $(\omega L)^2$ becomes larger and larger in the ratio R^2 to . So the root tends to ωL and thus to the numerator.

It can thus be seen that:

- at small frequencies there is no voltage U_2 at the output.
- at high frequencies $A = \frac {U_O}{U_I} = \rightarrow 1$, so the voltage at the output is equal to the voltage at the input.

Result:
The RL element shown here therefore only allows large frequencies to pass (= pass through) and small ones are filtered out.
The circuit corresponds to a **high pass**.

This can also be derived from understanding the components:

- At small frequencies, the current in the coil and thus the magnetic field changes only slowly. So only a negligibly small reverse voltage is induced. The coil acts like a short circuit at low frequencies.
- At higher frequencies, the current generated by U_I through the coil changes faster, the induced voltage $U_i = - dI / dt$ becomes large.
  As a result, the coil inhibits the current flow and a voltage drops across the coil.
- If the frequency becomes very high, only a negligible current flows through the coil - and hence through the resistor. The voltage drop at R thus approaches zero and the output voltage U_O tends towards U_I.

The transfer function can also be decomposed into amplitude response and frequency response. Often this plots are not given in with linear axis but:

- the amplitude response with a double logarithmic coordinate system and
- the phase response single logarithmic coordinate system.

By this, the course from low to high frequencies are easier to see. The following simulation in figure 3 shows the amplitude response and frequency response in the lower left corner.

Fig. 3: RL high pass filter

For further consideration, the equation of the transfer function $\underline{A} = \dfrac {\underline{U}_O^}{\underline{U}_I^}$ is to be rewritten so that it becomes independent of component values R and L.

This allows for a generalized representation. This representation is called **normalization**:

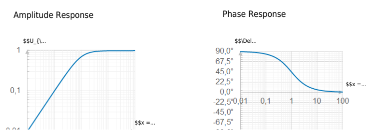
$$\begin{align*} \underline{A} = \frac {\underline{U}_O^}{\underline{U}_I^} = \frac {\omega L}{\sqrt{R^2 + (\omega L)^2}}\cdot e^{j\left(\frac{\pi}{2} - arctan \frac{\omega L}{R} \right)} \quad \xrightarrow{\text{normalization}} \quad \quad \underline{A}_{norm} = \frac {\omega L / R}{\sqrt{1 + (\omega L / R)^2}}\cdot e^{j\left(\frac{\pi}{2} - arctan \frac{\omega L}{R} \right)} = \frac {x}{\sqrt{1 + x^2}} \cdot e^{j\left(\frac{\pi}{2} - arctan x \right)} \end{align*}$$

$$\large{\underline{A} = \frac {\underline{U}_O^}{\underline{U}_I^} = \frac {\omega L}{\sqrt{R^2 + (\omega L)^2}}\cdot e^{j\left(\frac{\pi}{2} - arctan \frac{\omega L}{R} \right)} \quad \xrightarrow{\text{normalization}} \quad \quad \underline{A}_{norm} = \frac {\omega L / R}{\sqrt{1 + (\omega L / R)^2}}\cdot e^{j\left(\frac{\pi}{2} - arctan \frac{\omega L}{R} \right)} = \frac {x}{\sqrt{1 + x^2}} \cdot e^{j\left(\frac{\pi}{2} - arctan x \right)} }$$

This equation behaves quite the same as the one considered so far.

figure 4 shows the two plots. On the x-axis, $x = \omega L / R$ has been plotted as the normalization variable. This represents a weighted frequency.

Fig. 4: Amplitude and phase response of the RL high-pass filter



Here, too, the behavior determined in the limit value observation can be seen:

- at small frequencies ω (corresponds to small x), the amplitude response tends toward zero.
- At high frequencies, the ratio $U_O / U_I = 1$ is established.

Interesting in the phase response is the point $x = 1$.

- Further to the left of this point (i.e. at smaller frequencies) a tenfold increase of the frequency ω produces a tenfold increase of U_O / U_I.
- Further to the right of this point (i.e. at higher frequencies) $U_O / U_I = 1$ remains.

So this point marks a limit. Far to the left, the ohmic resistance is significantly greater the amount of impedance of the coil: $R \gg \omega L$. far to the right is just the opposite.

The point $x=1$ just marks the cut-off frequency.

It holds

$$\begin{align*} \underline{A}_{norm} = \frac{x}{\sqrt{1 + x^2}} \cdot e^{j\left(\frac{\pi}{2} - \arctan x \right)} = \frac{U_O}{U_I} \cdot e^{j\varphi} \quad \quad \left\{\begin{array}{l} x \ll 1 & \widehat{=}& \omega L \ll R &: \quad\quad \frac{U_O}{U_I}=x &, \varphi = \frac{\pi}{2} \, \widehat{=} \, 90° \\ x \gg 1 & \widehat{=}& \omega L \gg R &: \quad\quad \frac{U_O}{U_I}=1 &, \varphi = 0 \: \widehat{=} \, 0° \\ x = 1 & \widehat{=}& \omega L = R &: \quad\quad \frac{U_O}{U_I}=\frac{1}{\sqrt{2}} &, \varphi = \frac{\pi}{4} \, \widehat{=} \, 45° \end{array} \right. \end{align*}$$

Reminder:

- The **cut-off frequency** f_c for high-pass and low-pass filters is the frequency at which the ohmic resistance just equals the value of the impedance.
- The cut-off frequency separates a range in which the filter allows signals through from one in which they are suppressed (=blocked).
- At the cut-off frequency, the phase $\varphi = 45°$ and the amplitude $A = \frac{1}{\sqrt{2}}$.
- In German the cut-off Frequency is called Grenzfrequenz f_{Gr}

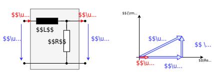
These statements apply to single-stage passive filters, i.e. one RL or one RC element.
Multistage filters are considered in circuit engineering.

The cut-off frequency in this case is given by:

$$\begin{align*} R &= \omega L \\ \omega_{c} &= \frac{R}{L} \\ 2 \pi f_{c} &= \frac{R}{L} \quad \rightarrow \quad \boxed{f_{c} = \frac{R}{2 \pi \cdot L}} \end{align*}$$

7.2.2 RL Low Pass

Fig. 5: Circuit, pointer diagram, and amplitude and phase response of RL low-pass filter



So far, only one variant of the RL element has been considered, namely the one where the output voltage $\underline{U}_O$ is tapped at the inductance.
Here we will briefly discuss what happens when the two components are swapped.

In this case, the normalized transfer function is given by:

$$\begin{align*} \underline{A}_{norm} = \frac {1}{\sqrt{1 + (\omega L / R)^2}}\cdot e^{-j \; \arctan \frac{\omega L}{R} } \end{align*}$$

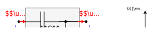
The cut-off frequency is again given by $f_{c} = \frac{R}{2 \pi \cdot L}$.

Fig. 6: RL low pass filter

7.3 RC Series Circuit

7.3.1 RC High Pass

Fig. 7: Circuit, pointer diagram, and amplitude and phase response of the RC high-pass filter



Now a voltage divider is to be constructed by a resistor R and a capacity C. Quite similar to the previous chapters, the transfer function can also be determined here.

Here results as normalized transfer function:

$$\begin{align*} \underline{A}_{norm} = \frac {\omega RC}{\sqrt{1 + (\omega RC)^2}}\cdot e^{\frac{\pi}{2}-j \; arctan (\omega RC) } \end{align*}$$

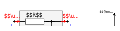
In this case, the normalization variable $x = \omega RC$. Again, the cut-off frequency is determined by equating R and the magnitude of the impedance of the capacitance:

$$\begin{align*} R &= \frac{1}{\omega_{c} C} \\ \omega_{c} &= \frac{1}{RC} \\ 2 \pi f_{c} &= \frac{1}{RC} \quad \rightarrow \quad \boxed{f_{c} =\frac{1}{2 \pi\cdot RC} } \end{align*}$$

Fig. 8: RC high pass filter

7.3.2 RC Low Pass

Fig. 9: Circuit, pointer diagram, and amplitude and phase response of RC low-pass filter



Again, the voltage at the impedance is to be used as the output voltage. This results in a low-pass filter.

Here results as normalized transfer function:

$$\begin{align*} \underline{A}_{norm} = \frac {1}{\sqrt{1 + (\omega RC)^2}}\cdot e^{-j \; arctan (\omega RC) } \end{align*}$$

Also, the cut-off frequency is given by $f_{c} =\frac{1}{2 \pi\cdot RC}$

Fig. 10: RC low pass filter