Block 16 - Ampère's Law and Magnetomotive Force (MMF)

Student Group

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Learning objectives

After this 90-minute block, you can

- …

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- …

90-minute plan

1. Warm-up (x min):
    1. ….
2. Core concepts & derivations (x min):
    1. …
3. Practice (x min): …
4. Wrap-up (x min): Summary box; common pitfalls checklist.

Conceptual overview

1. …

Core content

Generalization of the Magnetic Field Strength

So far, only the rotational symmetric problem of a single wire was considered in formula. I.e a current I and the length s of a magnetic field line around the wire was given to calculate the magnetic field strength H:

\begin{align*} \quad H_\varphi ={I\over{s}} = {{I}\over{2 \cdot \pi \cdot r}} \quad \Leftrightarrow \quad I = H_\varphi \cdot {s} \quad \quad \quad | \quad \text{applies only to the long, straight conductor} \end{align*}

Now, this shall be generalized. For this purpose, we will look back at the electric field.
For the electric field strength E of a capacitor with two plates at a distance of s and the potential difference U holds:

\begin{align*} U = E \cdot s \quad \quad | \quad \text{applies to plate capacitor only} \end{align*}

In words: The potential difference is given by adding up the field strength along the path of a probe from one plate to the other.
This was extended to the vltage between to points 1 and 2. Additionally, we know by Kirchhoff's voltage law that the voltage on a closed path is “0”.

\begin{align*} U_{12} &= \int_1^2 \vec{E} \; {\rm d}\vec{s} \\ U &= \oint \vec{E} \; {\rm d}\vec{s} =0 \\ \end{align*}

We can now try to look for similarities. Also for the magnetic field, the magnitude of the field strength is summed up along a path to arrive at another field-describing quantity.
Because of the similarity the so-called **magnetic potential difference V_m** between point 1 and 2 is introduced:

\begin{align*} V_m &= H \cdot s \quad \quad | \quad \text{applies to rotational symmetric problems only} \\ \end{align*}

\begin{align*} \boxed{ V_m = V_{m, 12} = \int_1^2 \vec{H} \; {\rm d}\vec{s} \\ V_m = \oint \vec{H} {\rm d}\vec{s} = \theta } \end{align*}

We need to take a loser look here. Any closed path in the static electric field leads to a potential difference of $U = \oint \vec{E} \; {\rm d}\vec{s} =0$.
BUT: closed paths in the static magnetic field leads to a magnetic potential difference which is **not mandatorily** 0! $ V_m = \oint \vec{H} {\rm d}\vec{s} = \theta$

Another new quantity is introduced: the **magnetic voltage θ**:

1. The magnetic voltage θ is the magnetic potential difference on a closed path.
2. Since the magnetic voltage θ is valid for exactly one turn along our single wire, θ is also equal to the current through the wire:
   \begin{align*} \theta = H \cdot s = I \quad \quad | \quad \text{applies only to the long, straight conductor} \end{align*}
3. The magnetic potential difference can take a fraction or a multiple of one turn and is therefore **not mandatorily** equal to I.
4. The magnetic voltage is generalized in the following box.

Notice:
The path integral of the magnetic field strength along an arbitrary closed path is equal to the free currents (= current density) through the surface enclosed by the path.

The magnetic voltage θ (and therefore the current) is the cause of the magnetic field strength.

This leads to the **Ampere's Circuital Law**

\begin{align*} \boxed{\oint_{s} \vec{H} {\rm d} \vec{s} = \theta } \end{align*}	The magnetic voltage θ can be given as • $\theta = I \quad \quad \quad \ $ for a single conductor • $\theta = N \cdot I \quad \:\; \, $ for a coil • $\theta = \sum_n \cdot I_n \quad$ for multiple conductors • $\theta = \iint_A \; \vec{S} {\rm d}\vec{A}$ for any spatial distribution (see block15)

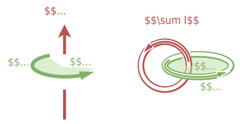
The unit of the magnetic voltage θ is **Ampere** (or **Ampere-turns**).

In the english literature the magnetic voltage is called **Magnetomotive force**

Notice:
${\rm d}\vec{s}$ and ${\rm d}\vec{A}$ in $\oint_{s} \vec{H} {\rm d} \vec{s} = \theta = \iint_A \; \vec{S} {\rm d}\vec{A}$ build a right-hand system.

1. Once the thumb of the right hand is pointing along ${\rm d}\vec{A}$, the fingers of the right hand show the correct direction for ${\rm d}\vec{s}$ for positive $\vec{H}$ and $\vec{S}$
2. Currents into the direction of the right hand's thumb count positive. Currents antiparallel to it count negative.

Fig. 1: Right hand rule



Recap of the fieldline images

longitudinal coil

Fig. 2: Magnetic field in a longitudinal coil

A longitudinal coil can be seen in figure 2.

The created field density of the coil can be derived from Ampere's Circuital Law

$$\begin{align*} \theta(t) &= \int & \vec{H}(t) \cdot {\rm d}\vec{s} \\ &= \int & \vec{H}_{\rm inner}(t) \cdot {\rm d}\vec{s} & + & \int \vec{H}_{\rm outer}(t) \cdot {\rm d} \vec{s} \\ &= \int & \vec{H}(t) \cdot {\rm d}\vec{s} & + & 0 \\ &= & {H}(t) \cdot l \\ \end{align*}$$

The magnetic field in a toroidal coil is often considered as homogenious in the inner volume, when the length l is much larger than the diameter: $l \gg d$.
With a given number N of windings, the magnetic field strength H is

toroidal coil

Fig. 3: Magnetic field in a toroidal coil

A toroidal coil has a donut-like setup. This can be seen in figure 3.
The toroidal coil is often defined by:

- The minor radius r: The radius of the circular cross-section of the coil.
- The major radius R: The distance from the center of the entire toroid (the center of the hole) to the center of the circular cross-section of the coil.

For reasons of symmetry, it shall get clear that the field lines form concentric circles.
Also the magnetic field strength H in a toroidal coil is often considered as homogenious, when $R \gg r$. With a given number N of windings, the magnetic field strength H is

\begin{align*} \theta = H \cdot l = N \cdot I \end{align*} \begin{align*} \boxed{H = {{N \cdot I}\over{l}}} \biggr | _\text{longitudinal coil} \end{align*}

\begin{align*} \theta = H \cdot 2\pi R = N \cdot I \end{align*} \begin{align*} \boxed{H = {{N \cdot I}\over{2\pi R}}} \biggr | _\text{toroidal coil} \end{align*}

Common pitfalls

- …

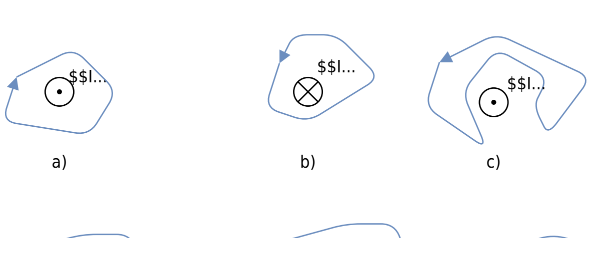
Exercises

Task 3.2.3 Magnetic Potential Difference

Fig. 3: different trajectories around current-carrying conductors

Result f)

f) $V_{\rm m,f} = 2~\rm A - 4.5~\rm A = - 2.5~\rm A$



Given are the adjacent closed trajectories in the magnetic field of current-carrying conductors (see figure 3). Let $I_1 = 2~\rm A$ and $I_2 = 4.5~\rm A$ be valid.

In each case, the magnetic potential difference $V_{\rm m}$ along the drawn path is sought.

Path

- The magnetic potential difference is given as the **sum of the current through the area within a closed path**.
- The direction of the current and the path have to be considered with the righthand rule.

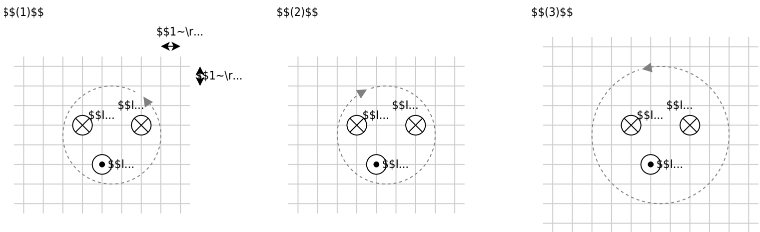
Exercise E13 Magnetic Voltage
(written test, approx. 6 % of a 120-minute written test, SS2021)

The following images show cross-sections of electrical cables.
A closed path is shown as a dashed line. The magnetic voltage θ on these paths shall be analyzed.
The following values are given for the currents:

- $I_1 = 5 {~\rm A}$
- $I_2 = 2 {~\rm A}$
- $I_3 = 1 {~\rm A}$
- $I_4 = 4 {~\rm A}$

Result

$$\begin{align*} \theta_{(1)} &= -4 {~\rm A} \\ \theta_{(2)} &= 0 {~\rm A} \\ \theta_{(3)} &= -5 {~\rm A} \\ \end{align*}$$



Specify which magnetic voltages $\theta_{(1)}$, $\theta_{(2)}$, and $\theta_{(3)}$ result. Note the direction of the path in each case!

Path

For the resulting current the direction of the path has to be considered with the right-hand rule:

- $I_{(1)} = +I_2 - I_1 - I_3 \quad \rightarrow \quad \theta_{(1)} = 2 {~\rm A} - 5 {~\rm A} - 1 {~\rm A} $
- $I_{(2)} = +I_3 + I_4 - I_1 \quad \rightarrow \quad \theta_{(2)} = 1 {~\rm A} + 4 {~\rm A} - 5 {~\rm A} $
- $I_{(3)} = +I_3 - I_4 - I_2 \quad \rightarrow \quad \theta_{(3)} = 1 {~\rm A} - 4 {~\rm A} - 2 {~\rm A} $

Exercise E1 Magnetic Potential

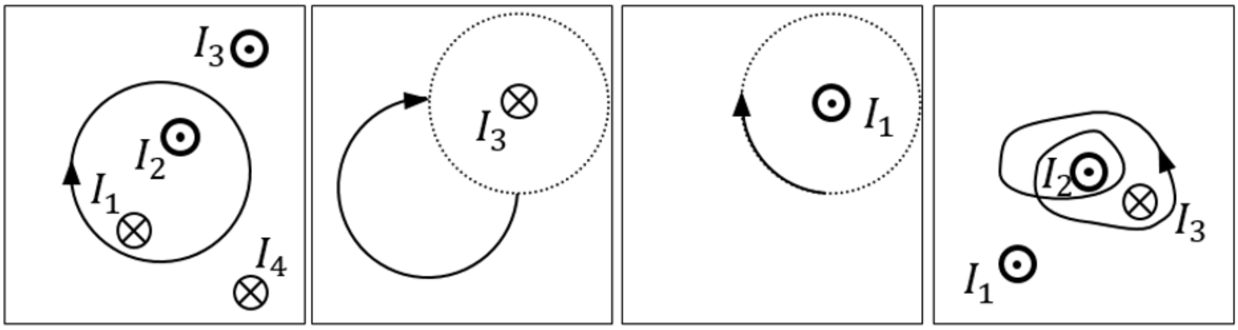
(written test, approx. 8 % of a 120-minute written test, SS2024)

Calculate the magnetic potential difference $V_\rm m$ for the following paths as shown by the solid lines.
Dotted lines are only for there for symmetry aspects!

The wires conduct the following currents:

- $|I_1|=2 ~\rm A$
- $|I_2|=5 ~\rm A$
- $|I_3|=11 ~\rm A$
- $|I_4|=7 ~\rm A$

Pay attention to the signs of the currents (given by the diagrams) and of the results!



1. 2. 3. 4.

Result

Based on the right-hand rule and the part of a full revolution the following results:

1. Task: $+I_1 - I_2 = -3 ~\rm A$

2. Task: $+{{1}\over{4}} I3 = 11/4 ~\rm A$ (it does not matter which way the path goes from the startpoint to the endpoint, as long as it has the same direction and number of revolutions)

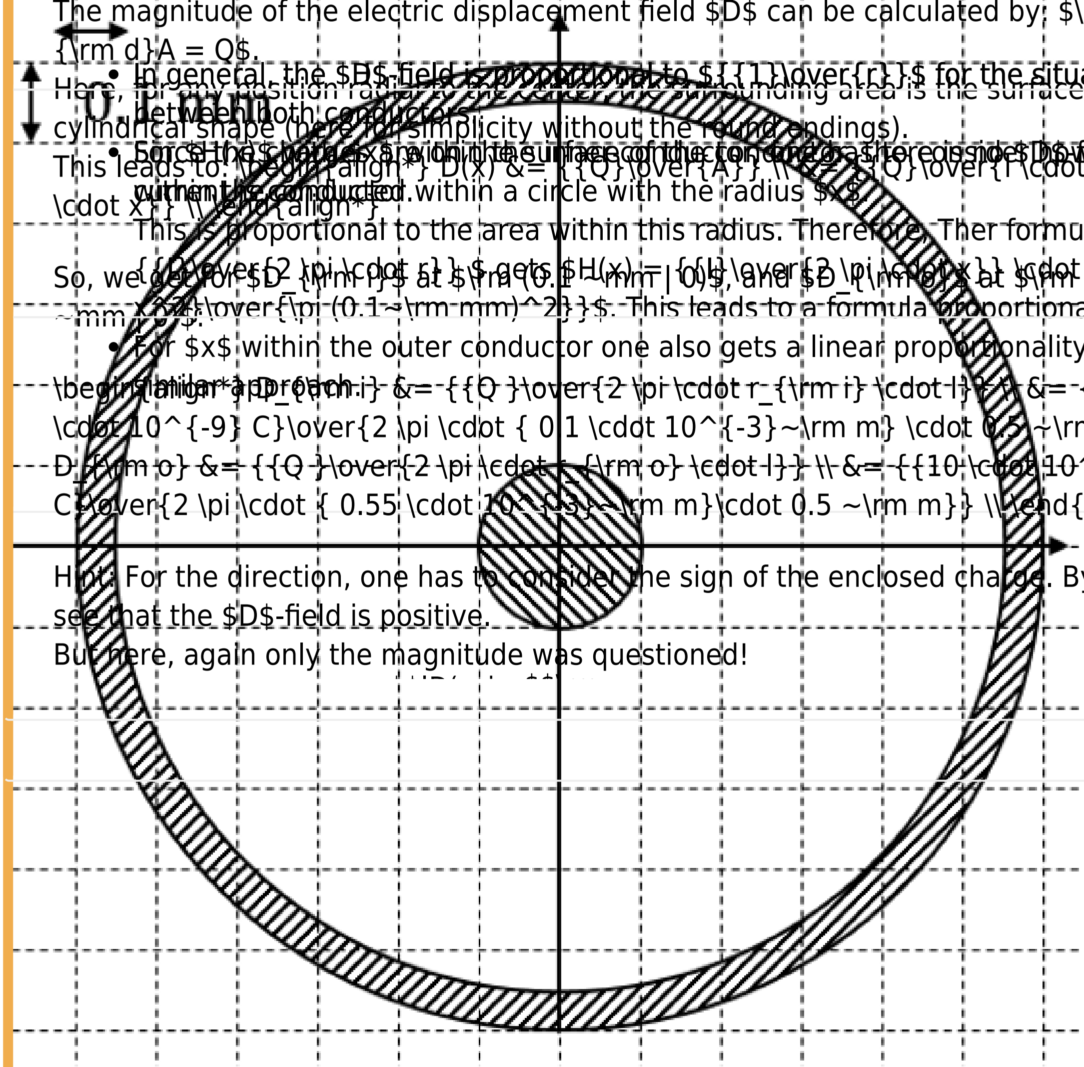
3. Task: $-{{1}\over{4}} I1 = -0.5 ~\rm A$

4. Task: $+2 \cdot I_2 - 1 \cdot I_3 = -1 ~\rm A$

Exercise E2 Fields of an coax Cable
(written test, approx. 12 % of a 120-minute written test, SS2024)

A. What is the magnitude of the electric displacement field D at $\rm (0.1 ~mm | 0)$ and $\rm (0.55 ~mm | 0)$? ... (0.6 ~mm | 0) ... cross-section ... the origin ... centre of the coax cable. Dimensions and labels for the diagram ... situation appears:

- Inner conductor: $+3.3 ~\rm mA$, $+10 ~\rm nC$ (current into the plane of the diagram)
- Outer conductor: $-3.3 ~\rm mA$, $ 0 ~\rm nC$ (current out of the plane of diagram)



Path

Result:

- for $(0.1 ~\rm mm | 0)$: $D_{\rm i} = 3.18... ~\rm µC/m^2$
- for $(0.55 ~\rm mm| 0)$: $D_{\rm o} = 0.585... ~\rm µC/m^2$

The magnitude of the electric displacement field D can be calculated by: $\int D {\rm d}A = Q$.
Here, for any position radius r, the surrounding area is the surface of a cylindrical shape (here for simplicity without the round endings).
This leads to: \begin{align*} D(x) &= {{Q}\over{A}} \\ &= {{Q}\over{l \cdot 2\pi \cdot x}} \\ \end{align*}

So, we get for $D_{\rm i}$ at $\rm (0.1 ~mm | 0)$, and $D_{\rm o}$ at $\rm (0.55 ~mm | 0)$:

\begin{align*} D_{\rm i} &= {{Q }\over{2 \pi \cdot r_{\rm i} \cdot l}} \\ &= {{10 \cdot 10^{-9} C}\over{2 \pi \cdot { 0.1 \cdot 10^{-3}~\rm m} \cdot 0.5 ~\rm m}} \\ D_{\rm o} &= {{Q }\over{2 \pi \cdot r_{\rm o} \cdot l}} \\ &= {{10 \cdot 10^{-9} C}\over{2 \pi \cdot { 0.55 \cdot 10^{-3}~\rm m}\cdot 0.5 ~\rm m}} \\ \end{align*}

Hint: For the direction, one has to consider the sign of the enclosed charge. By this, we see that the D-field is positive.
But here, again only the magnitude was questioned!

Path

- for $(0.1 ~\rm mm | 0)$: $H_{\rm i} = 5.25... ~\rm A/m$
- for $(0.55 ~\rm mm| 0)$: $H_{\rm o} = 0.955... ~\rm A/m$

- In general, the H-field is proportional to ${{1}\over{r}}$ for the situation between both conductors.
- For x values within the surface of the conductor, there is only the current within a circle with the radius x. This is proportional to the area within this radius. Therefore, Ther formula $H = {{I}\over{2 \pi \cdot r}} $ gets $H(x) = {{I}\over{2 \pi \cdot x}} \cdot {{\pi x^2}\over{\pi (0.1~\rm mm)^2}}$. This leads to a formula proportional to x.
- For x within the outer conductor one also gets a linear proportionality with a similar approach.

1. What is the magnitude of the magnetic field strength H at $\rm (0.1 ~mm | 0)$ and $\rm (0.55 ~mm | 0)$?

Path

The magnitude of the magnetic field strength H can be calculated by: $H = {{I}\over{2 \pi \cdot r}} $
So, we get for $H_{\rm i}$ at $\rm (0.1 ~mm | 0)$, and $H_{\rm o}$ at $\rm (0.55

> ~mm | 0)$:
>
> \begin{align*} H_{\rm i} &= {{I}\over{2 \pi \cdot r_{\rm i}}} \\ &= {{+3.3 A}\over{2 \pi \cdot { 0.1 \cdot 10^{-3}~\rm m}}} \\ H_{\rm o} &= {{I}\over{2 \pi \cdot r_{\rm o}}} \\ &= {{+3.3 A}\over{2 \pi \cdot { 0.55 \cdot 10^{-3}~\rm m}}} \\ \end{align*}
>
> Hint: For the direction, one has to consider the right-hand rule. By this, we see that the H-field on the right side points downwards.
> Therefore, the sign of the H-field is negative.
> But here, only the magnitude was questioned!

Embedded resources

Explanation (video): …